Generalized Functions- Exercise 4

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1. We need to show that for all $x \in \mathbb{Q}$

$$| x |_\infty \cdot \prod_p | x |_p = 1$$

This is true because by factoring we have that for any $x \in \mathbb{Q}$ $| x |_\infty = | m |_\infty / | n |_\infty = \prod_p p^{i_p}$ where $i_p \in \mathbb{Z}$ and for almost all primes $i_p = 0$. On the other hand $| x |_p = | m |_p / | n |_p = p^{-i_p}$. This implies that $\prod_p | x |_p = | x |_\infty^{-1}$, as required.

2. (a) We need to show that $B_\varepsilon(a)$ is closed, or equivilantly that $B_\varepsilon(a)^c$ is open. This is true becasue if $x \in B_\varepsilon(a)^c$ and $y \in B_\varepsilon(x)$, then $d(y, a) = max(d(x, y), d(x, a)) \geq \varepsilon$ (because $d(x, y) \neq d(x, a)$). So $B_\varepsilon(x) \subseteq B_\varepsilon(a)^c$ and therefore $B_\varepsilon(a)^c$ is open.

(b) We need to show that every point in $B_\varepsilon(a)$ is its center, i.e that for any $x \in B_\varepsilon(a)$ we have $B_\varepsilon(x) = B_\varepsilon(a)$. By symmertry it suffices to show that $B_\varepsilon(x) \subseteq B_\varepsilon(a)$, which is true because for any $y \in B_\varepsilon(x)$ we have $d(y, a) \leq max(d(x, a), d(x, y)) \leq \varepsilon$.

(c) This is true because the p-adic norm only takes values in the countable set $\{p^m \mid m \in \mathbb{Z}\}$, and therefore $B_\varepsilon(0) = B_{p^m}(0)$ for some $m \in \mathbb{Z}$.

3. (a) We need to show that the cantor set is homeomorphic to $\mathbb{Z}_p$. The

first space is homeomorphic to$\{0,1\}^{\mathbb{N}}$, and the latter to $\{0,1,..,p-1\}^{\mathbb{N}}$ ($\mathbb{Z}_p$ and $\{0,1,..p-1\}^{\mathbb{Z}}$ are in obvious bijection, ans this bijection is a homeomorphism because the topology on $\mathbb{Z}_p$ is induced by the inverse limit of discrete spaces $\mathbb{Z}/\mathbb{Z}_{p^n}$), so it suffices to show that these two spaces are homeomorphic. Note that each of these spaces is metrizable (as a countable products of metrizable spaces), compact (by Tychonov's theorem), totally disconnected (as a product of totally disconnected spaces) and perfect. We'll prove the following, more general claim: Any two compact, totally disconnected perfect metric spaces are homeomorphic. Let M be such a space. First we show that M is homeomorphic to a certain inverse limit of discrete finite subsets. First note that for any $n \in N$, any cover of M has a finite refinement consisting of disjoint clopen sets of diameter$\leq 1/n$. Indeed, this follows from question 6 and from the fact that open balls generate the topology of M. Now consider the sequence $V_1, V_2, ...$ of finite covers, such that V_i consists of clopen disjoint sets of diameter$\leq 1/i$ and V_{i+1} is a refinement of V_i. We give all of these sets the discrete topology and consider the inverse system $(\{V_n\}, i_n)$ with respect to the maps $i_n : V_n \to V_{n-1}$ where $i_n(U)$ is the unique (from disjointness) element of V_{n-1} containing U. Let $L = \underset{\leftarrow}{lim} V_n$ be the inverse limit of this system. We claim that M is homeomorphic to L: indeed, consider the map g that takes $x \in M$ to the sequence $\{U_n(x)\} \in L$ where $U_n(x)$ is the unique element of the cover V_n containing x. Note that this map is injective because the diameter of sets in V_n is $\leq 1/n$, and surjective by the finite intersection property (as the sets are closed as well as open). g is clearly continuous as well, because $g^{-1}(\{\{X_n\} \in L; X_m = U \in V_m\} = U$ which is open. As a continuous bijection from a compact haussdorf space, g is a homeomorphism. Now let M_1, M_2 be two compact, totally disconnected perfect metric spaces. We want to build a sequences of covers $\{A_n\}, \{B_n\}$ of M_1, M_2 with inverse systems $(\{A_n\}, i_n^A), (\{B_n\}, i_n^B)$ (defined as before) whose inverse limits are homeomorphic. We do this as follows: Start with two arbitrary sequences $\{A_n\},\{B_n\}$.We first claim that given an open subset U of either space covered by n disjoint clopen sets $K_1,...,K_n$, one can find a disjoint clopen refinement consisting of m sets, for any $m\geq n$. Indeed, K_1 isn't a singleton because it's open, so it's not connected. Therefore we have $K_1=A\cup B$ for some disjoint clopen sets A,B . Repeating this process $n-m$ times gets us the required refinement. Now if $|A_1|<|B_1|$ we refine decompose some set in A_1 into $|B_1|-|A_1|+1$ disjoint clopen sets to offset the difference. We'll abuse notation and call these covers A_1,B_1 as well. So assume $|A_1|=|B_1|$and take some bejection $f_1:|A_1|\rightarrow|B_1|$. Now we refine A_2,B_2 in the same way (again abusing notation) so that $|A_2|=|B_2|$ and for each $x\in A_1$we have $|i_2^A(x)^{-1}|=|i_2^B(f(x))^{-1}|$(i.e x is decomposed into as many parts in A_2 as $f(x)$ is in B_2). We then take a bijection $f_2:A_2\rightarrow B_2$ such that $f_1\circ i_2^A=i_2^B\circ f_2$. Continuing in this fashion we get a sequence of bijections $\{f_n:|A_n|\rightarrow|B_n|\}$such that $f_n\circ i_{n-1}^A=i_{n+1}^B\circ f_{n+1}$. These maps then induce a map $f:\underset{\leftarrow}{lim}A_n\rightarrow\underset{\leftarrow}{lim}B_n$ by $f(\{a_n\})=\{f(a_n)\}$ (where $a_n\in A_n$). We claim that f is our required homeomorphism: indeed, f is clearly a bijection because f_n are for all n, and continuity is obvious because f_n are all continous (being maps from discrete spacces). The continuity of the inverse is also clear (because f^{-1} is induced by $f_n^{-1}:|B_n|\rightarrow|A_n|$ in the same way). We conclude that M_1, and M_2 are homeomorphic, and in particular so are $\{0,1\}^{\mathbb{N}}$ and $\{0,1,..,p-1\}^{\mathbb{N}}$.

(b) Note that $\mathbb{Q}_p$ is homeomorphic to the countable disjoint union of the sets $\mathbb{Q}_p(a_1,...,a_k)=\{x\in Q_p:x=....\textbf{.}a_1..a_k\}$ (numbers with a given finite p-adic expansion after the p-cimal point). These sets are clearly all homeomorphic to $\mathbb{Z}_p$, which in turn is homemorphic to the cantor set $C=\{0,1\}^{\mathbb{N}}$. Therefore it suffices to show that $C-\{x_0\}$ is homeomorphic to a disjoint union of cantor sets $\bigsqcup\limits_{i\in N}C_i$. But this is easy, because $\{0,1\}^{\mathbb{N}}-\{x_0\}=\bigsqcup\limits_{i\in N}\left\{x\in\{0,1\}^{\mathbb{N}}:\underset{j\in N}{min}(x(j)\neq x_0(j))=i\right\}$

and this is a disjoint union of sets that are again clearly hoemeomorphic to C.

(c) $\mathbb{Q}_p, \mathbb{Z}_p$ are both of cardinality c (being countable unions of sets of cardinality c) , and both are totally disconnected (so their connected componenets are singletons).

4. Consider $X = \{0,1\}^{\mathbb{R}}$ with the product topology (where every copy of$\{0,1\}$ is discrete). X is totally disconnected as a product of totally disconnected spaces. For $x_1 \neq x_2$, take $a \in R$ such that $x_1(a) \neq x_2(a)$. Then $U_1 = \{x \in X; x(a) = x_1(a)\}$, $U_2 = \{x \in X; x(a) = x_2(a)\}$ are disjoint open neighborhoods of x_1, x_2respectively. So X is haussdorf. Finally, by Tychonoff's theorem X is compact (and in particular locally compact). Thus X is a compact ℓ space. Now consider the subset $U = X - \{x_0\}$ for some point $x_0 \in X$. We claim that U isn't countable at ∞. Indeed, if this were the case then any open cover of U would have a countable subcover, becuase $U{=}\bigcup_{n=1}^{\infty} K_n$ where $K_n$are compact, and each one of those sets has a finite subcover. However the open sets $B_a = \{x \in X; x(a) \neq x_0(a)\}$for $a \in R$ cover U, and this cover has no countable subcover (by the uncountability of R).

5. We need to show that any non-archimedian field F is an ℓ space, i.e a totally disconnected, locally compact haussdorf space. We have local compactness by definition. For $x \neq y$ we have $\mid x - y \mid = a > 0$, by the triangle inequality $B_{a/2}(x) \cap B_{a/2}(y) = \emptyset$. So F is haussdorf. Note that by question 2 all balls in F are clopen (the exact same argument applies), and we have $\bigcap\limits_{a>0} B_a(x) = \{x\}$, so the quasi componenets (i.e the intersection of all clopen sets containing a point, which coincide with connected components for locally compact Haussdorf spaces) are all singletons and F is totally disconnected, and we are done.

6. (a) We need to show that any ℓspace X has a basis of open compact sets. Let $x \in X$, and let U be an open subset of X with $x \in U$. As a Haussdorf,

locally-compact space X has a local basis of open sets with compact closure, so there's an open set s.t $x \in V \subseteq U$ with $cl(V)$ compact. Note that $cl(V)$ is totally disconnected as a subspace of a totally disconnected space, and because it's compact it is totally seperated, so the intersection of all clopen subsets of $cl(V)$ is $\{x\}$, i.e for every point $y \neq x$ there's a clopen subset of $cl(V)$, which we denote by $B(y)$, that doesn't contain y. This means that its complement in $cl(V)$, which we denote by $C(y)$, is a clopen subset that does contain y. Now $V, \{C(y); x \neq y \in cl(V)\}$.together form an open cover of $cl(V)$. By compactness there's a finite subcover $V, C(y_1), ..., C(y_n)$. Then $x \in \bigcap_{i=1}^{n} B(y_i) \subseteq V$. We claim that this subset is compact and open in X. Indeed, this subset is open in V, which is open in X, so it's open in X as well. The subset is compact in X because it's a closed subset of the compact space $cl(V)$. So we have found a compact open subset of U containing x. Thus X has a basis of compact open sets.

(b) We need to show that given a compact subset $K \subseteq X$, any open cover of K has a finite refinement of disjoint open compact sets covering K. Let $K \subseteq \bigcup_{\beta \in I} V_\beta$ be an open cover. By (a) we can take a refinement $\{U_\alpha; \alpha \in J\}$ of compact-open sets. Note that because X is Haussdorf these sets are all closed as well. Now by the compactness of K there exists a finite subcover $U_1, ..., U_n$. Now consider the cover $B_1 = U_1$, $B_i = U_i \bigcap_{j=1}^{j=i-1} U_j^c$ for $i \geq 2$. These sets are open as finite intersections of open sets, compact as closed subsets of compact sets ($B_i \subseteq U_i$), and disjoint by construction. So we've found a finite refinement consisting of compact, disjoint sets.